Integration by trigonometric substitution

1. The three common trigonometric substitution are the restricted sine, restricted tangent and restricted secant. Ans: True

2. Typically trigonometric substitutions are used for problems that involve radical expression s? Ans: True

3. In integration by substitution, when $\sqrt{a^2+u^2}$ the value of u is $a\tan\theta$. Ans: True

4. In integration by substitutions, when $\sqrt{a^2-u^2}$ the value of du is $a\sin^2\theta\, d\theta$. Ans: False, $a\cos\theta\, d\theta$

5. In integration by substitutions, when $\sqrt{u^2-a^2}$ the value of du is $a\cos^2\theta\, d\theta$. Ans: False, $a\sec\theta\tan\theta\, d\theta$

6. $\int \frac{\sqrt{16-x^2}}{x^2}dx$ Ans: $-\frac{\sqrt{16-x^2}}{x}-\sin^{-1}\left(\frac{x}{4}\right)+C$

7. $\int \sqrt{1-6y^2}\,dy$ Ans: $\frac{1}{2\sqrt{6}}\left[\sin^{-1}\left(\sqrt{6}y\right)+\sqrt{6}y\sqrt{1-6y^2}\right]+C$

8. $\int \cos x\sqrt{4+25\sin^2 x\ dx}$ Ans: $=\frac{2\ln\left(\left|\sqrt{25\sin^2 x+4}+5\sin x\right|\right)}{5}+\sin x\sqrt{\frac{25\sin^2(x)}{4}+1}+C$

9. $\int_1^3 2x^5\sqrt{4+16x^2}dx$ Ans: $=2546.1589$

10. $\int_{-6}^{-4}\frac{3}{x^3\sqrt{x^2-16}}dx$ Ans: $=-0.015301$

Solution:

6. $\int \frac{\sqrt{16-x^2}}{x^2}dx$

$a^2=16, \qquad a=4$

$x=4\sin\theta, \qquad dx=4\cos\theta d\theta$

$=\int\frac{\sqrt{16-x^2}}{x^2}$

$=\int\frac{\sqrt{16-(4\sin\theta)^2}}{(4\sin\theta)^2}\cdot 4\cos\theta\, d\theta$

$=\int\frac{\sqrt{16-16\sin^2\theta}}{16\sin^2\theta}\cdot 4\cos\theta\, d\theta$

7. $\int \sqrt{1-6y^2}\,dy$

$y=\frac{1}{\sqrt{6}}\sin\theta$

$\sqrt{1-6y^2}$

$=\sqrt{1-\sin^2\theta}$

$=\sqrt{1-\cos^2\theta}$

$=|\cos\theta|$

$= \int \frac{\sqrt{16(1-\sin^2\theta)}}{4\sin^2\theta}\cdot\cos\theta\, d\theta$

$= \int \frac{4\sqrt{\cos^2\theta}}{4\sin^2\theta}\cdot \cos\theta\,d\theta$

$= \int \frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\,d\theta$

$= \int\frac{\cos^2\theta}{\sin^2\theta}$

$= \int \frac{1-\sin^2\theta}{\sin^2\theta}d\theta$

$= \int[\frac{1}{\sin^2\theta} - \frac{\sin^2\theta}{\sin^2\theta}]$

$= \int[\csc^2\theta - 1]\,d\theta$

$= -\cot\theta - \theta + C$

$= -\frac{\sqrt{16-x^2}}{x} - \sin^{-1}\left(\frac{x}{4}\right) + C$

$x = 4\sin\theta,$

$\frac{x}{4} = \sin\theta$

$\sin^{-1}\frac{x}{4} = \sin^{-1}(\sin\theta)$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$\tan\theta = \frac{x}{\sqrt{16-x^2}}$

$\cot\theta = \frac{1}{\tan\theta}$

$dy = \frac{1}{\sqrt{6}}\cos\theta\,d\theta$

$\int\sqrt{1-6y^2}$

$= \int\cos\theta\,(\frac{1}{\sqrt{6}}\cos\theta)\,d\theta$

$= \frac{1}{\sqrt{6}}\int\cos^2\theta\ \ d\theta$

$= \frac{1}{\sqrt{6}}\int\frac{1}{2}[1+\cos(2\theta)]\,d\theta$

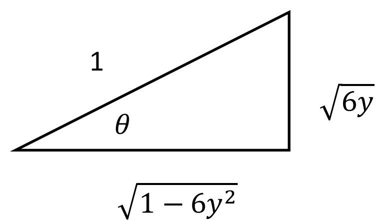
$= \frac{1}{2\sqrt{6}}\left[\theta+\frac{1}{2}\sin(2\theta)\right]+C$

$= \frac{1}{2\sqrt{6}}[\theta+\sin(\theta)\cos(\theta)]+C$

$\sin\theta=\sqrt{6} \quad\rightarrow\quad \theta=\sin^{-1}(\sqrt{6}y)$

$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{6y}}{1}$

$\cos\theta = \sqrt{1-6y^2}$



$= \frac{1}{2\sqrt{6}}\left[\sin^{-1}(\sqrt{6y}) + \sqrt{6y}\sqrt{1-6y^2}\right]+C$

8. $\int \cos x\sqrt{4+25\sin^2 x}\,dx = \int\sqrt{4+25\,[\sin x]^2}$

$u = \sin x \qquad du = \cos x\,dx$

$dx = \frac{1}{\cos x}du$

$\int\sqrt{25u^2+4}du$

$u = \frac{2\tan v}{5} \qquad v = \arctan\left(\frac{5u}{2}\right)$

$\mathrm{du} = \frac{2\sec^2(v)}{5}$

$= \int\frac{2\sec^2(v)\sqrt{4\tan^2(v)+4}}{5}dv$

$= \frac{4}{5}\int\sec^3(v)dv$

9. $\int_1^3 2x^5\sqrt{4+16x^2}dx$

$4\int x^5\sqrt{4x^2+1}\,dx$

$u = 4x^2+1 \qquad du = 8x \qquad dx = \frac{1}{8x}du$

$= \frac{1}{128}\int(u-1)^2\sqrt{u}du$

$= \frac{1}{128}\int\left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + \sqrt{u}\right)du$

$= \int u^{\frac{5}{2}}\,du - 2\int u^{\frac{3}{2}}du + \int\sqrt{u}du$

$= \frac{2u^{\frac{7}{2}}}{7} - \frac{4u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}$

$\frac{1}{128}\int(u-1)^2\sqrt{u}du$

$\int \sec^3(v)dv$

$= \frac{\sec v \tan v}{2} + \frac{1}{2}\int \sec v\, dv$

$\int \sec v\, dv$

$= \ln(\tan v + \sec v)$

$= \frac{\sec v \tan v}{2} + \frac{1}{2}\int \sec v\, dv$

$= \frac{\ln(\tan v + \sec v)}{2} + \frac{\sec v \operatorname{tuan} v}{2}$

$= \frac{4}{5}\int \sec^3(v)dv$

$= \frac{2\ln(\tan v + \sec v)}{5} + \frac{2 \sec v \tan v}{5}$

Undo substitution: $u = \arctan(\frac{5u}{2})$

$\tan(\arctan\frac{5u}{2}) = \frac{5u}{2}$

$\sec(\arctan\frac{5u}{2}) = \sqrt{\frac{25u^2}{4} + 1}$

$= \frac{2\ln\left(\sqrt{\frac{25\sin^2(x)}{4}+1}+\frac{5\sin x}{2}\right)}{5} + \sin x\sqrt{\frac{25\sin^2(x)}{4} + 1} + C$

$= \frac{2\ln(|\sqrt{25\sin^2 x+4}+5\sin x|)}{5} + \sin x\sqrt{\frac{25\sin^2(x)}{4} + 1} + C$

10. $\int_{-7}^{-5} \frac{3}{x^3\sqrt{x^2-25}}dx$

$= 3\int \frac{1}{x^3\sqrt{x^2-25}}dx$

$u = \sqrt{x^2 - 25}$ $\quad du = \frac{x}{\sqrt{x^2-25}}$ $\quad dx = \frac{\sqrt{x^2-25}}{x}$

$= \int \frac{1}{(u^2+25)^2}du$

Apply reduction formula:

$$\int \frac{1}{(au^2+b)^n}du = \frac{2n-3}{2b(n-1)}\int \frac{1}{(au^2+b)^{n-1}}du + \frac{u}{2b(n-1)(au^2+b)^{n-1}}$$

$a = 1, b = 25, n = 2$

$= \frac{u^{\frac{7}{2}}}{448} - \frac{u^{\frac{5}{2}}}{160} + \frac{u^{\frac{3}{2}}}{192}$

$= \frac{(4x^2+1)^{\frac{7}{2}}}{448} - \frac{(4x^2+1)^{\frac{5}{2}}}{160} + \frac{(4x^2+1)^{\frac{3}{2}}}{192}$

$4\int x^5\sqrt{4x^2+1}\,dx$

$= \frac{(4x^2+1)^{\frac{7}{2}}}{112} - \frac{(4x^2+1)^{\frac{5}{2}}}{40} + \frac{(4x^2+1)^{\frac{3}{2}}}{48}$

$= \frac{(4x^2+1)^{\frac{3}{2}}(30x^4-6x^2+1)}{210} + C$

$= 2(\frac{2377.37^{\frac{3}{2}}}{420} - \frac{5^{\frac{5}{2}}}{84})$

$= 2546.1589$

$$= \frac{u}{50(u^2+25)} + \frac{1}{50}\int \frac{1}{u^2+25}\,du$$

$$\int \frac{1}{u^2+25}\,du$$

$v = \frac{u}{5}$ $\qquad dv = \frac{1}{5}$ $\qquad du = 5dv$

$$= \int \frac{5}{25v^2+25}\,dv$$

$$= \frac{1}{5}\int \frac{1}{v^2+1}\,dv$$

$$= \frac{1}{5}\int \frac{1}{v^2+1}\,dv$$

$$= \frac{\arctan v}{5}$$

$$= \frac{u}{50(u^2+25)} + \frac{1}{50}\int \frac{1}{u^2+25}\,du$$

$$= \frac{u}{50(u^2+25)} + \frac{\arctan(\frac{u}{5})}{250}$$

$$= \frac{\arctan(\frac{\sqrt{x^2-25}}{5})}{250} + \frac{\sqrt{x^2-25}}{50x^2}$$

$$3\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$$

$$= \frac{3\arctan(\frac{\sqrt{x^2-25}}{5})}{250} + \frac{3\sqrt{x^2-25}}{50x^2}$$

$$= 3\left(\frac{49\arcsin\left(\frac{5}{7}\right)-10\sqrt{6}}{12250} - \frac{\pi}{500}\right)$$

$$= \frac{294\arcsin\left(\frac{5}{7}\right)-147\pi-10\cdot 6^{\frac{3}{2}}}{24500}$$

$$= -0.015301$$